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On Non-Singular Generalised Dynamical Formalisms

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Abstract

This work shows that a certain class of classical dynamical formalisms, characterised by non-singular Lie structures more general than the usual (Poisson) one, are derivable from ordinary constrained dynamical formalisms. As a consequence, the Lie brackets considered are special cases of suitably chosen Dirac brackets. Both unconstrained and constrained generalised dynamical formalisms are considered. The relations of our results with the problem of constructing classical analogues of generalised quantum systems are stressed.

1. *Introduction*

The important role that classical models have played in the development of the formal structure of the usual quantum theory is well known. In fact, when a classical analogue exists it already contains many of the properties of its quantum partner. In this particular circumstance lies the heuristic importance of the quantisation schemes proposed since the origins of quantum mechanics by Dirac (1950, 1958, 1964), for Bose-like systems, and later enlarged by Droz-Vincent (1966) and Franke & Kálnay (1970) to cover Fermi-like systems. Hopefully a similar situation may exist when dealing with quantum systems whose formal structure is more complex than that of the Bose or Fermi systems. We have in mind, for instance, Green's Parasystems. All this suggests the convenience of studying the possibility of constructing classical analogues of such general systems. By a classical analogue we mean here, as before, a classical model described by a set of coordinates (analytic functions of time) where, besides the usual commutative product, a Lie product has been defined. This Lie product must (a) define, in the conventional way, the dynamics of the system and (b) through some quantisation rule reproduce the formal structure of the original quantum system.

Recently Kálnay (1972) has given a step towards the solution of the program sketched above for the generalised quantum systems to which increasing interest has been devoted in later years, i.e. the above-mentioned Green's Parasystems. In Kálnay's work the Lie and Jordan products, through which the Green's trilinear algebra is defined, are realised classically by means of a skew-symmetric and a symmetric Dirac bracket respectively. This choice seems natural because (a) classical analogues of the Bose and Fermi type can be constructed with the aid of Dirac brackets (Kálnay & Ruggeri, 1972; Kálnay, 1972), a fact which suggests its eventual relevance for Parasystems; and (b) Dirac brackets are the more general classical brackets which have been studied with some detail. There are no *a priori* reasons, however, for discarding some other types of Lie brackets in the description of classical analogues of Parasystems or even more general quantum systems. The aim of this and, hopefully, future works is to throw some light on the problem by choosing an approach which differs from that of Kálnay. We study generalised dynamical formalisms and look for relations between them and ordinary constrained systems. For the latter the natural Lie structure is the Dirac bracket or some generalised form of it (Bergmann & Goldberg, 1955).

After stating our notations and conventions we consider in Section 3 a restricted class of dynamical formalism which generalises the ordinary canonical one by replacing the Poisson bracket by a more general bracket. In Section 4 we analyse the new elements which arise when we introduce integrable constraints between the coordinates. In both cases we show that the generalised dynamical formalism can be embedded into an ordinary one with a double number of variables. The main results are summarised in Section 5.

2. *Notations and Conventions*

The name ordinary dynamical formalism will be given in this work to both the conventional Hamiltonian theory of classical mechanics and to the theory developed by Dirac (1950, 1964) to treat systems for which the phase-space variables are not independent. We call *generalised dynamical formalisms* those classical dynamical formalisms obtained from the ordinary ones by replacing the Poisson bracket by a more general Lie bracket which will be specified below in Section 3.

Coordinates and Indices

The canonical variables of the formalism will be denoted by α^μ, where μ, as any Greek index, takes the values 1, 2, . . ., $2N$. Also

$$\alpha \equiv (\alpha^1, \alpha^2, \ldots, \alpha^{2N}) \tag{2.1}$$

In Section 3 momenta canonically conjugated to α^μ will be introduced; we use for it the symbol π_μ, and

$$\pi \equiv (\pi_1, \pi_2, \ldots, \pi_{2N}) \tag{2.2}$$

The *ordinary* canonical formalism whose variables are $\{(\alpha, \pi)\}$ will be called stretched formalism. The summation convention, as well as the abbreviations $\partial_\mu \equiv (\partial/\partial\alpha^\mu)$ and $\partial^\mu \equiv (\partial/\partial\pi_\mu)$, will be used systematically.

Brackets

The bracket $\{\ ,\ \}_-^\Gamma$ denotes the Lie bracket defined by

$$\{F, G\}_-^\Gamma = \Gamma^{\mu\nu}\,\partial_\mu F\,\partial_\nu G \tag{2.3}$$

The tensor $\{\Gamma^{\mu\nu}\}$ satisfies the relations (3.3) to (3.5). The bracket $\{\ ,\ \}_-^A$ is the ordinary Poisson bracket of the stretched theory

$$\{\mathscr{F}, \mathscr{G}\}_-^A = \partial_\mu \mathscr{F}\,\partial^\mu \mathscr{G} - \partial^\mu \mathscr{F}\,\partial_\mu \mathscr{G} \tag{2.4}$$

$\mathscr{F}$ and $\mathscr{G}$, as any other script symbols, denote functions which take values in the phase space $\{(\alpha, \pi)\}$.

The bracket $\{\ ,\ \}_-^{\Gamma*}$ denotes the generalised Dirac bracket introduced in Section 4. $\{\ ,\ \}_-^{A*}$ is the ordinary Dirac bracket of the stretched theory. It is defined by:

$$\{\mathscr{F}, \mathscr{G}\}_-^{A*} = \{\mathscr{F}, \mathscr{G}\}_-^A - \{\mathscr{F}, \theta_a\}_-^A \mathscr{C}^{ab}\{\theta_b, \mathscr{G}\}_-^A \tag{2.5}$$

where $\{\theta_a;\ a = 1, 2, \ldots, N_\theta\}$ is the irreducible set of second-class constraints appropriate to each case. This set is such that:

$$\mathscr{C}^{ab}\{\theta_b, \theta_c\}_-^A = \delta^a{}_c \tag{2.6}$$

for all a and c.

Finally, the square bracket $[\ ,\ ]_-$ will be reserved for the commutator.

3. *Unconstrained Generalised Dynamical Formalism*

By an unconstrained generalised dynamical formalism we mean a dynamical theory in which the state of the system is fully characterised by $2N$ independent variables $\{\alpha^\mu\}$ (the canonical variables) which evolve in time according to

$$\dot{\alpha}^\mu = \{\alpha^\mu, H\}_-^\Gamma \tag{3.1}$$

In these equations H (the Hamiltonian) is a function of the α's and the time, and $\{\ ,\ \}_-^\Gamma$ is the bracket defined by†

$$\{F, G\}_-^\Gamma = \Gamma^{\mu\nu}\,\partial_\mu F\,\partial_\nu G \tag{3.2}$$

for any pair of functions of the α's and also, eventually, the time. We will suppose that this bracket has the following properties:

(a) It defines a Lie product in the space of (infinitely differentiable) functions of α. That is, the following relations hold:

(a.1) Antisymmetry

$$\{F, G\}_-^\Gamma = -\{G, F\}_-^\Gamma \tag{3.3}$$

† The brackets defined by equations (3.2) to (3.5) have been considered frequently in the literature. See, for example, Martin (1959) and Mukunda & Sudarshan (1968).

(a.2) Jacobi Identity

$$\{F,\{G,J\}_-^{\Gamma}\}_-^{\Gamma} + \{J,\{F,G\}_-^{\Gamma}\}_-^{\Gamma} + \{G,\{J,F\}_-^{\Gamma}\}_-^{\Gamma} = 0 \tag{3.4}$$

(b) It is non-singular in the sense that $\det\|\Gamma^{\mu\nu}\| \neq 0$. The inverse matrix will be denoted by $\|\Gamma_{\mu\nu}\|$:

$$\Gamma_{\mu\lambda}\,\Gamma^{\lambda\nu} = \delta_\mu{}^\nu \tag{3.5}$$

Equations (3.3) and (3.4) are respectively equivalent to

$$\Gamma^{\mu\nu} = -\Gamma^{\nu\mu} \tag{3.6}$$

and

$$\Gamma^{\lambda\mu}\,\partial_\mu\,\Gamma^{\nu\rho} + \Gamma^{\nu\mu}\,\partial_\mu\,\Gamma^{\rho\lambda} + \Gamma^{\rho\mu}\,\partial_\mu\,\Gamma^{\lambda\nu} = 0 \tag{3.7}$$

It is now easy to show that equations (3.5) and (3.7) lead to

$$\partial_\lambda\,\Gamma_{\mu\nu} + \partial_\nu\,\Gamma_{\lambda\mu} + \partial_\mu\,\Gamma_{\nu\lambda} = 0 \tag{3.8}$$

This last relation shows the existence of a (quasi)vector $\{f_\mu\}$ such that

$$\Gamma_{\mu\nu} = \partial_\mu\,f_\nu - \partial_\nu\,f_\mu \tag{3.9}$$

and which is obviously defined up to a gauge transformation

$$f_\mu \to f'_\mu = f_\mu + \partial_\mu\,G \tag{3.10}$$

Equation (3.1) can now be shown to follow from a variational principle of the ordinary type (Martin, 1959). In fact by equations (3.2), (3.5) and (3.9) we can write equation (3.1) as

$$(\partial_\mu\,f_\nu - \partial_\nu\,f_\mu)\,\dot{\alpha}^\nu = \partial_\mu\,H \tag{3.11}$$

These are just the Euler–Lagrange equations corresponding to the family of Lagrangians

$$\mathscr{L}_G = f_\mu\,\dot{\alpha}^\mu - H + dG/dt \tag{3.12}$$

where again G is arbitrary.†

Consider now the ordinary canonical formalism derived from $\mathscr{L}$. The momentum conjugated to α^μ is:

$$\pi_\mu(\alpha,\dot{\alpha}) \equiv (\partial\mathscr{L}/\partial\dot{\alpha}^\mu)(\alpha,\dot{\alpha}) = f_\mu(\alpha) \tag{3.13}$$

and the Hamiltonian

$$\mathscr{H}(\alpha,\pi) \equiv \dot{\alpha}^\mu\,\pi_\mu - \mathscr{L} = H(\alpha) \tag{3.14}$$

Equation (3.13) shows the existence of $2N$ primary constraints‡

$$\chi_\mu \equiv \pi_\mu - f_\mu \approx 0 \qquad \mu = 1, 2, \ldots, 2N \tag{3.15}$$

The canonical equations are then of Dirac's type, i.e.:

$$\dot{\alpha}^\mu = \partial^\mu\,\mathscr{H} + \lambda^\nu\,\partial^\mu\,\chi_\nu \tag{3.16a}$$

$$-\dot{\pi}_\mu = \partial_\mu\,\mathscr{H} + \lambda^\nu\,\partial_\mu\,\chi_\nu \tag{3.16b}$$

† The indeterminacies expressed by equations (3.10) and (3.12) are, of course, not independent. They are related to the possibility of performing canonical transformations with respect to the bracket $\{\ ,\ \}_-^{\Gamma}$. In the following we shall take $G = 0$ and $\mathscr{L}_0 = \mathscr{L}$.

‡ We follow the definitions and conventions established by Dirac (1964).

where the $2N$ λ's are additional variables. Due to equations (3.14), (3.15) and (3.16a) they coincide with the velocities

$$\lambda^\nu = \dot{\alpha}^\nu \tag{3.17}$$

By substitution of equations (3.13) and (3.17) into equation (3.16b) we recover equation (3.11) and then also equation (3.1). In this way a generalised canonical formalism is embedded in an ordinary one by stretching the phase space of the system. We have now $4N$ variables instead of $2N$ but the number of independent ones is the same as before due to the constraints of equations (3.15).

There are no secondary constraints here, the consistence equations $\dot{\chi}_\mu \approx 0$, $\mu = 1, 2, \ldots, 2N$, serving only to determine the λ's as in equation (3.17).

The above results suggest the existence of a close relation between the algebraic structures naturally associated with the generalised formalism on the one hand and with the ordinary constrained formalism on the other. These structures are respectively defined by the brackets $\{\,,\,\}_-^\Gamma$ and $\{\,,\,\}_-^{A*}$. To show this relation, note first that each one of the χ's is second class because

$$\{\chi_\mu, \chi_\nu\}_-^A = \partial_\mu f_\nu - \partial_\nu f_\mu = \Gamma_{\mu\nu} \tag{3.18}$$

Moreover, the set $\{\chi_\mu\}$ is an irreducible set of constraints from which we can construct the Dirac bracket. Obviously we have (cf. equation (2.6))

$$\mathscr{C}^{\mu\nu} = \Gamma^{\mu\nu} \tag{3.19}$$

Now take two functions F and G of α. We have:

$$\{F, G\}_-^A = 0; \qquad \{F, \chi_\mu\}_-^A = \partial_\mu F, \text{ etc.} \tag{3.20}$$

and then equations (3.19), (3.20) and (2.5) lead us to

$$\{F, G\}_-^{A*} = \Gamma^{\mu\nu}\,\partial_\mu F\,\partial_\nu G = \{F, G\}_-^\Gamma \tag{3.21}$$

This last equation shows that any non-singular Lie bracket of the type (3.2) is in fact deducible from an ordinary Dirac bracket. The corresponding set of irreducible second-class constraints is given by $\{\chi_\mu \equiv \pi_\mu - f_\mu;\ \mu = 1, 2, \ldots, 2N\}$.†

Let us make two remarks concerning the relations just found:

(a) Equations (3.1) and (3.21) imply that for any function F of α

$$dF/dt = \{F, H\}_-^{A*} \tag{3.22}$$

In the general case equation (3.22) must be written as (Dirac, 1964)

$$dF/dt \approx \{F, H'\}_-^{A*} \tag{3.23}$$

† Cawley (1969) has proved previously that the ordinary Hamiltonian theory can be deduced from a stretched Dirac theory and that the quantisation is unaltered if the Poisson bracket is replaced by the Dirac bracket. Some of our denominations and procedures were suggested by that work.

where H' is the extended Hamiltonian in the sense of Dirac. The stronger version of equation (3.22) is a consequence of the very simple functional form of the constraints of the stretched formalism.

(b) For the Bose-like quantisation procedure of systems described by a generalised dynamical formalism the following rule has been proposed as a natural extension of the ordinary (Dirac's) one ($\hbar = 1$)

$$i\{,\}_-^{\Gamma} \to [,]_- \tag{3.24}$$

By virtue of equation (3.21) this rule is a special case of the following one (Dirac, 1950, 1964)

$$i\{,\}_-^{*} \to [,]_- \tag{3.25}$$

We stress the fact that in equations (3.24) and (3.25) brackets appear which are respectively associated to a generalised formalism and to an ordinary, but stretched, one.

V. *Constrained Generalised Dynamical Formalism*

The simplest extension of the ordinary Hamiltonian theory is obtained by assuming that a certain number of integrable constraints hold between the phase-space variables. The corresponding theory has been analysed in detail by Dirac (1950, 1964). This theory suggests that consideration be given to constrained generalised dynamical formalisms which are just characterised by the existence of, say, N_1 *a priori* constraints

$$\phi_i(\alpha) = 0 \qquad i = 1, \ldots, N_1 \tag{4.1a}$$

written also as

$$\phi_i \approx 0 \qquad i = 1, \ldots, N_1 \tag{4.1b}$$

Following Dirac, the equations of motion are postulated to be given by

$$\dot{\alpha}^\mu = \Gamma^{\mu\nu}(\partial_\nu H + u^i \partial_\nu \phi_i) \tag{4.2a}$$

Here $\Gamma^{\mu\nu}$ satisfies equations (3.5) to (3.7). Equation (4.2a) can also be written as

$$\dot{\alpha}^\mu = \{\alpha^\mu, H\}_-^{\Gamma} + u^i\{\alpha^\mu, \phi_i\}_-^{\Gamma} \tag{4.2b}$$

In equations (4.2) the u's are not determined *a priori* but are necessary for a complete dynamical description.

For the following considerations it is useful to summarise here the main aspects of a dynamical theory whose elements are equations (4.1) and (4.2). Let us follow Dirac's approach.†

(i) Besides the primary constraints (4.1) other constraints, called secondary, may appear by elimination of the u's from the set of consistence equations $\dot{\phi}_i \approx 0$, $i = 1, \ldots, N_1$, i.e., from

$$\{\phi_i, H\}_-^{\Gamma} + u^j\{\phi_i, \phi_j\}_-^{\Gamma} \approx 0 \tag{4.3}$$

† See, for instance, Dirac (1964). In the following we shall not prove any proposition for which the proofs given by Dirac are also valid here.

The new constraints must also satisfy consistence equations like equation (4.3) and so on. This procedure leads us finally to:

(i(a)) A closed set of, say, N_c constraints which we denote by $\{\Psi_m; m = 1, \ldots, N_1, \ldots, N_c\}$.

(i(b)) N_c consistence equations. A subset of these equations permits one to obtain generally some of the u's, while the rest remains completely arbitrary.

(ii) The theory of constrained systems can also be formulated, in an equivalent form, in terms of non-singular linear combinations of the Ψ's. By means of these combinations a new complete set of constraints $\{(\psi_p, \theta_a); a = 1, 2, \ldots, N_\theta; p = 1, \ldots, (N_c - N_\theta)\}$ can be constructed with the following properties

(ii(a)) The ψ's are *Γ first class.* This means that

$$\{\psi_p, \psi_{p'}\}_-^{\Gamma} \approx 0 \qquad \text{for all } p \text{ and } p' \tag{4.4a}$$

and also

$$\{\psi_p, \theta_a\}_-^{\Gamma} \approx 0 \qquad \text{for all } p \text{ and } a \tag{4.4b}$$

(ii(b)) $\{\theta\}$ is an irreducible set of Γ *second-class* constraints. This means that for any a there exists at least an a' such that

$$\{\theta_a, \theta_{a'}\}_-^{\Gamma} \not\approx 0 \tag{4.5}$$

and, moreover, it is not possible to obtain a Γ first-class constraint by taking linear combinations of the θ's. In this case, just as Dirac has proved, it can be shown that the matrix $\|\{\theta_a, \theta_b\}_-^{\Gamma}\|$ is non-singular. Let us call $C_\Gamma = \|C_\Gamma^{ab}\|$ the inverse matrix:

$$C_\Gamma^{ab}\{\theta_b, \theta_c\}_-^{\Gamma} = \delta^a{}_c \tag{4.6}$$

(iii) The Lie bracket naturally associated to a constrained system with Γ second-class constraints is the Dirac bracket constructed from $\{\,,\}_-^{\Gamma}$, i.e.:

$$\{F, G\}_-^{\Gamma *} = \{F, G\}_-^{\Gamma} - \{F, \theta_a\}_-^{\Gamma} C_\Gamma^{ab}\{\theta_b, G\}_-^{\Gamma} \tag{4.7}$$

This means, in particular, that when quantising our system according to the conventional scheme inconsistencies will be avoided if the rule

$$i\{\,,\}_-^{\Gamma *} \to [\,,]_- \tag{4.8}$$

is used instead of the rule of equation (3.24).

The dynamical equations (4.2) can now be shown to be also deducible from a variational principle. As can be expected, they are the Euler–Lagrange equations corresponding to the Lagrangians (3.12) with equations (4.1) as constraints and the u's as undetermined coefficients, i.e.:

$$(\partial\mathscr{L}/\partial\alpha^\mu) - d(\partial\mathscr{L}/\partial\dot{\alpha}^\mu)/dt = u^i(\partial\phi_i/\partial\alpha^\mu) \tag{4.9}$$

As in the previous section, an ordinary canonical formalism can be constructed from equation (4.9). We are led again to

$$\chi_\mu \equiv \pi_\mu - f_\mu \approx 0 \tag{4.10}$$

$$\mathscr{H}(\alpha, \pi) = H(\alpha) \tag{4.11}$$

As we have here two sets of primary constraints: $\{\phi\}$ and $\{\chi\}$, the Dirac equations are:

$$\dot{\alpha}^\mu = \partial^\mu H + \lambda^\nu \partial^\mu \chi_\nu + u^i \partial^\mu \phi_i \qquad (4.12a)$$

$$-\dot{\pi}^\mu = \partial_\mu H + \lambda^\nu \partial_\mu \chi_\nu + u^i \partial_\mu \phi_i \qquad (4.12b)$$

Consider now the consistence equations $\dot{\chi}_\mu \approx 0$ and $\dot{\phi}_i \approx 0$. The first one is:

$$\{\chi_\mu, H\}_-^A + \lambda^\nu\{\chi_\mu, \chi_\nu\}_-^A + u^i\{\chi_\mu, \phi_i\}_-^A \approx 0 \qquad (4.13)$$

which due to the relations

$$\{\chi_\mu, H\}_-^A = -\partial_\mu H \qquad (4.14a)$$

$$\{\chi_\mu, \chi_\nu\}_-^A = \Gamma_{\mu\nu} \qquad (4.14b)$$

$$\{\chi_\mu, \phi_i\}_-^A = -\partial_\mu \phi_i \qquad (4.14c)$$

and to equations (3.5), is equivalent to:

$$\lambda^\mu \approx \Gamma^{\mu\nu}(\partial_\nu H + u^i \partial_\nu \phi_i) \qquad \mu = 1, 2, \ldots, 2N \qquad (5.15)$$

The original equations of motion are recovered by comparing equation (4.5) with equation (4.12a). Equation (4.15) shows that, as could be expected, the λ's are determined if the u's are. In order to restrict the u's we need the remaining consistence equations. As

$$\{\phi_i, H\}_-^A = \{\phi_i, \phi_j\}_-^A = 0 \qquad (4.16a)$$

we have simply:

$$\dot{\phi}_i = \lambda^\mu \partial_\mu \phi_i \approx 0 \qquad i = 1, \ldots, N_1 \qquad (4.16b)$$

Comparing finally equations (4.15) and (4.16b) we arrive at

$$\Gamma^{\mu\nu}(\partial_\mu \phi_i \partial_\nu H + u^j \partial_\mu \phi_i \partial_\nu \phi_j) \approx 0 \qquad (4.17)$$

which is equivalent to equation (4.3).

In summary, the generalised dynamical theory described by any of equations (4.2) and the set of N_c constraints $\{\Psi\}$ is equivalent to an ordinary dynamical theory with the set of $2N + N_c$ constraints:

$$\{(\chi_\mu, \Psi_m); \mu = 1, 2, \ldots, 2N; m = 1, \ldots, N_c\} \qquad (4.18)$$

Let us examine the structure of the set (4.18). Due to equations (4.14b) and (4.14c) all constraints of that set are second class. Nevertheless, *the set* $\{(\chi, \Psi)\}$ *is not irreducible* as the following considerations show. Let us define

$$\Psi'_m = \Psi_m + \chi_\mu \Gamma^{\mu\nu} \partial_\nu \Psi_m \approx 0 \qquad m = 1, \ldots, N_c \qquad (4.19)$$

The results (4.20) below can be easily proved by using elementary properties of the Poisson bracket and equations (4.10), (4.14b) and (4.14c):

$$\{\Psi'_m, \chi_\mu\}_-^A \approx 0 \qquad (4.20a)$$

$$\{\Psi'_m, \Psi'_n\}_-^A \approx \{\Psi_m, \Psi_n\}_-^\Gamma \qquad m, n = 1, \ldots, N_c; \mu = 1, 2, \ldots, 2N \qquad (4.20b)$$

Equations (4.20) show that if Ψ is Γ first class (resp. Γ second class) then Ψ' is first class (resp. second class) in the stretched formalism. Thus, by means of the linear combinations (4.19) the Ψ constraints are classified according to its character in the generalised formalism.

Let $\{\theta\}$ be an irreducible set of Γ second-class constraints and define the following set of $2N+N_\theta$ constraints:

$$\{\xi_1, \ldots, \xi_{2N}, \ldots, \xi_{(2N+N_\theta)}\} \equiv \{\chi_1, \ldots, \chi_{2N}, \theta'_1, \ldots, \theta'_{N_\theta}\} \tag{4.21}$$

This last one is an irreducible set of second-class constraints for the stretched formalism. To see this note that:

$$\{\xi_u, \xi_v\}_-{}^A = \Gamma_{uv} \qquad \text{if } 1 \leqslant u, v \leqslant 2N \tag{4.22a}$$

$$\{\xi_u, \xi_v\}_-{}^A = 0, \qquad \text{if } u \leqslant 2N \text{ and } v > 2N \text{ or } u > 2N \text{ and } v \leqslant 2N \tag{4.22b}$$

$$\{\xi_u, \xi_v\}_-{}^A = \{\theta_u, \theta_v\}_-{}^\Gamma \quad \text{if } u, v > 2N \tag{4.22c}$$

on the phase space restricted by equation (4.10). It is now easy to verify that the $(2N+N_\theta) \times (2N+N_\theta)$ matrix

$$\mathscr{C} \equiv \begin{Vmatrix} \Gamma & 0 \\ 0 & C_\Gamma \end{Vmatrix} \tag{4.23}$$

with $\Gamma \equiv \|\Gamma^{\mu\nu}\|$ and $C_\Gamma \equiv \|C_\Gamma^{ab}\|$, satisfies:

$$\mathscr{C}^{uv}\{\xi_v, \xi_w\}_-{}^A = \delta^u{}_w \quad u, v, w = 1, 2, \ldots, (2N+N_\theta) \tag{4.24}$$

The natural Lie bracket of the stretched formalism is the Dirac bracket of equation (2.5). If we restrict ourselves to functions of the α, we have:

$$\{F, G\}_-{}^A = 0 \tag{4.25a}$$

$$\{F, \xi_u\}_-{}^A = \delta_u{}^\alpha \partial_\alpha F \qquad \text{if } u \leqslant 2N \tag{4.25b}$$

$$\{\xi_u, F\}_-{}^A = \delta_{u-2N}{}^a\{\theta'_a, F\}_-{}^A \qquad \text{if } u > 2N, \text{ etc.} \tag{4.25c}$$

According to equations (4.23) and (4.25) we can write:

$$\{F, G\}_-{}^{A^*} = \Gamma^{\mu\nu}\partial_\mu F \partial_\nu G - \{F, \theta'_a\}_-{}^A C_\Gamma^{ab}\{\theta'_b, G\}_-{}^A \tag{4.26}$$

Furthermore, substituting the identities

$$\begin{aligned} \{F, \theta'_a\}_-{}^A &= \{F, \theta_a + \chi_\mu \Gamma^{\mu\nu}\partial_\nu \theta_a\}_-{}^A \\ &= \{F, \chi_\mu\}_-{}^A \Gamma^{\mu\nu}\partial_\nu\theta_a \\ &= \{F, \theta_a\}_-{}^\Gamma, \qquad \text{etc.} \end{aligned} \tag{4.27}$$

into equation (4.26), we finally get:

$$\{F, G\}_-{}^{A*} = \{F, G\}_-{}^{\Gamma*} \tag{4.28}$$

This last equation completes the proof that a constrained generalised dynamical formalism can be embedded in an ordinary (Dirac type) stretched dynamical theory. In particular the extension of the usual procedure of quantisation, expressed by equation (4.8), is again contained in Dirac's rule (3.25).

5. *Summary and Discussion*

This work has been devoted to show the interrelations which exist between certain generalised dynamical formalisms and Dirac's theory for constrained systems. An interesting corollary is that certain non-singular Lie brackets are special cases of suitably chosen Dirac brackets. This fact suggests that this last structure is a convenient starting point for constructing classical analogues of quantum systems whose structure is more complex than the usual ones. As has been mentioned in the Introduction, this conjecture has been previously made by Kálnay (1972) for the interesting case of Green's Parasystems. In spite of the partial success of this approach, which is supported to some extent by our results, more general Lie structures must not be discarded *a priori*. Some of these may arise from the analysis of some classes of *singular* generalised dynamical formalisms.†

In the Parasystems case there is another problem to consider, which is the construction of a classical Jordan algebra. In Kálnay's work this was done by means of the symmetric Dirac bracket introduced previously by Franke & Kálnay (1970). This approach can also be followed here, by introducing for instance a symmetric partner for the bracket (4.7). However, it is not clear to the author that such classical realisations of the Jordan algebra are to be considered as satisfactory if we want to preserve, at the classical level, some of the more simple properties of quantum Parasystems.‡

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† We are referring to such structures as the 'generalised Dirac bracket' introduced by Bergmann & Goldberg (1955). This bracket may play an important role in the analysis of a class of singular dynamical formalism which is now under consideration by the author.

‡ As an example, we mention that, following Kálnay's procedure, the classical Bose algebra is not a realisation of the classical Parabose algebra, as opposed to what happens at the quantum level.